

Periodic orbits in outer billiard

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Abstract

It is shown that the set of 4-period orbits in outer billiard with piecewise smooth convex boundary has an empty interior, provided that no four corners of the boundary form a parallelogram.

1 Introduction

The study of periodic orbits has been always important in the field of Hamiltonian dynamics and classical billiard is one of the early examples of a Hamiltonian dynamical system. This system was introduced by G.D. Birkhoff, see *e.g.* [3, 4], who also showed that classical billiard with smooth convex boundary possesses at least two periodic orbits of each (p, q) type, see *e.g.* [10], for the proof of this result.

More recently, additional interest for the study of sets of periodic orbits came from the spectral theory of Laplace operator on bounded domains. V. Ivrii showed [9] that the so-called Weyl's asymptotics of distribution of large eigenvalues of the Laplacian (with the Dirichlet or Neumann boundary conditions) holds, provided periodic orbits of the corresponding classical system constitute the set of zero measure. Then, in order to fill the gap in the proof of what is sometimes called Weyl's conjecture [19], one has to demonstrate that the union of periodic orbits is a set of measure zero in the billiard phase space. Therefore, in contrast to Birkhoff theorem and its generalizations (see *e.g.* [5]), here one has to study the upper bound on the number (or rather measure) of periodic orbits.

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Establishing that periodic orbits have zero measure turns out to be surprisingly hard for periodic orbits of arbitrary period. The only case which can be easily dealt with is the case of two period orbits. Indeed, since the segments of two period orbits must be normal to the boundary, then from each boundary point there can emanate only one 2-period orbit. These orbits then form at most one-parameter family which obviously has zero measure in the two dimensional phase space.

The case of three period orbits is already non-trivial. It was solved by Rychlik, see [13]. The proof involved some symbolic calculations that were later removed in [14]. A much simpler proof relying on Jacobi's fields appeared in [20]. Later in [18] this result for three period orbits was extended to higher dimensional billiards. All these proofs have been obtained by first demonstrating that there are no open sets of periodic orbits and then verifying that the sets of positive measure do not exist either. The second part of the proof is relatively easy as the sets of positive measure have density points whose neighborhoods are "almost" open sets.

More recently, another approach based on the theory of exterior differential systems (EDS) has been proposed to study open sets of periodic orbits by Landsberg, Baryshnikov and the second author [1], [11]. For systematic exposition of the EDS theory along with many applications, see *e.g.* [8]. Similar billiard formulation has been independently developed by Török [17]. The EDS approach gives a systematic proof in the three period case and reduces 4-period case to the study of zeros of certain system of polynomials. Unfortunately, the system is too hard to resolve even with the aid of symbolic calculations (at least by direct use of Maple or Mathematica).

In this paper, we consider a closely related system of outer (or dual) billiard, which is another popular model in Hamiltonian dynamics. Originally introduced by Bernhard Neumann, the outer billiard was popularized by Jürgen Moser [12] and others as a model stability problem. See also the survey article [16] for more information and recent results on outer billiards.

The dynamics of the outer billiard is defined in the exterior of a convex boundary $\Gamma \in \mathbb{R}^2$ as follows: draw a line L through a point $z_0 = (x_0, y_0) \in \mathbb{R}^2$ tangent to Γ in, say, the counterclockwise direction. Find a point $z_1 = (x_1, y_1) \in L$ and such that the tangency point is dividing the segment $|z_0, z_1|$ in half. The induced map $P : (x_0, y_0) \rightarrow (x_1, y_1)$ defines the outer (dual) billiard dynamics. The map is not well defined for lines for which the tangency point is not unique. However such lines are countable and therefore, the outer billiard map is not well defined on at most a set of zero measure [7]. The exterior of the boundary can be then considered as a phase space and the invariant measure is given by the area form $\mu = dx \wedge dy$.

The natural extension of the conjecture for classical billiard is that periodic orbits in outer billiard constitute the set of zero measure (except, may be, for some special boundaries, see section 3.5). While the outer billiard does not have such significance for the spectral asymptotics problem, our hope is that this study will help resolve related problem for the classical billiard. We have been also motivated by a recent article by Genin and Tabachnikov [6] which (among other results) contains a proof that the set of 3-period orbits in outer billiard has an empty interior.

In this article, we study the set of 4-period orbits in outer billiards. Our main result is contained in

Theorem 1.1 *Let $\Gamma \subset \mathbb{R}^2$ be a piecewise smooth convex closed curve. Assume that no four corners of Γ form a parallelogram. Then the set of 4-period orbits in the outer billiard has an empty interior.*

This theorem follows from Theorem 2.1 on certain properties of the exterior differential system associated with the outer billiard. We recall that in the EDS approach, instead of asking which outer billiard boundaries possess open sets of periodic orbits, one studies which 2-parameter families of quadrilaterals can (or cannot) be orbits in an outer billiard. More precisely, we search for 2-dimensional disks of quadrilaterals in the space of all quadrilaterals. These 2-dimensional disks must satisfy certain differential relations.

In the next section, we “translate” the problem in the language of exterior differential systems. This EDS corresponds to the Birkhoff distribution in the case of classical billiard [1, 2] and to the dual Birkhoff distribution in case of outer billiard [16]. Then we find the solutions of that EDS under some nondegeneracy conditions. As it turns out, for each nondegenerate quadrilateral there exists only one EDS solution, which corresponds to a 2-parameter family of 4-period orbits in an outer billiard. We verify that there are no other solutions by proving that Frobenius type integrability condition does not hold.

2 EDS associated with outer billiard

Since the set of initial conditions for which the billiard map is not well defined has zero measure, we restrict ourselves to the complementary subset where the map is well defined. Below, we always assume that the outer billiard map is well defined.

We start with the proposition which establishes relation between open sets of n -periodic orbits and integral submanifolds in an associated exterior differential systems (see [1], [2] or [11] for a related statement for the classical billiard).

Proposition 2.1 *Suppose that there exists an open set Q of n -periodic orbits in the outer billiard phase space for the billiard with a convex piecewise smooth boundary $\Gamma \subset \mathbb{R}^2$. Then there exists a 2-dimensional disk in the space of n -gons $M^2 \subset \mathbb{R}^{2n}$ such that*

$$\theta^i|_{TM^2} = 0,$$

where $i \in \mathbb{Z}/n\mathbb{Z}$ and

$$\theta^i = \frac{1}{2}(y_i - y_{i+1})d(x^i + x^{i+1}) - \frac{1}{2}(x_i - x_{i+1})d(y^i + y^{i+1}).$$

The following nondegeneracy conditions hold: all points are different

$$(x_i, y_i) \neq (x_j, y_j) \text{ if } i \neq j$$

and no three consecutive points belong to the same line

$$(x_{i-1} - x_i)(y_i - y_{i+1}) \neq (x_i - x_{i+1})(y_{i-1} - y_i).$$

The area form $dx^i \wedge dy^i \neq 0$ on M^2 for all i .

Proof: Let M^2 be the set of periodic orbits in \mathbb{R}^{2n} , the space of n -gons. Any one-parameter family of periodic orbits $z_i(t) = (x_i(t), y_i(t)) \in M^2$, where $z_i \in \mathbb{R}^2$ and $i \in \mathbb{Z}/n\mathbb{Z}$, must satisfy the tangency condition (middle point of any segment cannot move in the normal direction to the segment)

$$\frac{d}{dt} \left(\frac{z_{i+1} + z_i}{2} \right) = \lambda(z_{i+1} - z_i),$$

where $\lambda \in \mathbb{R}$. This relation implies that θ^i must vanish.

In a sufficiently small neighborhood of each tangency point, the boundary is either smooth and convex or it has a corner, therefore M^2 is an embedding of Q in \mathbb{R}^{2n} , the space of n -gons. Since Q is an open set, then $dx^1 \wedge dy^1 \neq 0$. Verification of nondegeneracy conditions is straightforward. \square

By this proposition, it remains to find all two dimensional integral submanifolds in the exterior differential system generated by θ^i and satisfying the above nondegeneracy conditions. We will refer to such integral manifolds as nondegenerate.

Next theorem gives the local description of the two dimensional integral manifolds in the outer billiard EDS.

Theorem 2.1 *For any nondegenerate convex quadrilateral there exists a unique nondegenerate connected integral manifold containing the quadrilateral. This manifold is given by the quadrilaterals whose middle points coincide with those of the original quadrilateral.*

3 Proof of the theorem 2.1

3.1 New coframe

Supplementing θ^i with ω^i

$$\omega^i = \frac{1}{2}(y_i - y_{i+1}) d(x^i - x^{i+1}) - \frac{1}{2}(x_i - x_{i+1}) d(y^i - y^{i+1}), \quad (1)$$

we obtain a coframe $\{\theta^i, \omega^i\}_{i=1}^{i=n}$. It is easy to check that these $2n$ forms are linearly independent on an open dense subset of \mathbb{R}^{2n} : $\{(x_i, y_i) \neq (x_j, y_j), (x_{i-1} - x_i)(y_i - y_{i+1}) \neq (x_i - x_{i+1})(y_{i-1} - y_i), i, j = 1, 2, \dots, n\}$ using the following identities which can be directly verified

$$(y_i - y_{i+1})dx^{i+1} - (x_i - x_{i+1})dy^{i+1} = \theta^i - \omega^i \quad (2)$$

$$(y_{i+1} - y_{i+2})dx^{i+1} - (x_{i+1} - x_{i+2})dy^{i+1} = \theta^{i+1} + \omega^{i+1}. \quad (3)$$

Note that the determinant of the above linear system

$$\Delta_{i,i+1} = \begin{vmatrix} y_i - y_{i+1} & -(x_i - x_{i+1}) \\ y_{i+1} - y_{i+2} & -(x_{i+1} - x_{i+2}) \end{vmatrix} = \begin{vmatrix} x_i - x_{i+1} & y_i - y_{i+1} \\ x_{i+1} - x_{i+2} & y_{i+1} - y_{i+2} \end{vmatrix}$$

does not vanish by the nondegeneracy conditions (consecutive points do not belong to the same line).

This determinant $\Delta_{i,i+1}$ has a clear geometric meaning. It is the double area of the triangle with the vertices (x_i, y_i) , (x_{i+1}, y_{i+1}) , and (x_{i+2}, y_{i+2}) (assuming the vertices are enumerated counterclockwise). The total area of the n -gon is an integral for the system [6]. Indeed, adding the forms, we obtain

$$\sum_{i=1}^n \theta^i = \frac{1}{2} d \left(\sum_{i=1}^n (y_i x_{i+1} - y_{i+1} x_i) \right), \quad (4)$$

where the sum on the right handside is the total area of the n -gon. Therefore, for the quadrilateral ($n = 4$) we have

$$\Delta_{1,2} + \Delta_{3,4} = 2S = \text{constant} \quad (5)$$

$$\Delta_{2,3} + \Delta_{4,1} = 2S = \text{constant}, \quad (6)$$

where S is the area of the quadrilateral.

Solving (2–3), we obtain

$$dx^{i+1} = \frac{1}{\Delta_{i,i+1}} ((x_{i+1} - x_{i+2})\omega^i + (x_i - x_{i+1})\omega^{i+1}) \quad (7)$$

$$dy^{i+1} = \frac{1}{\Delta_{i,i+1}} ((y_{i+1} - y_{i+2})\omega^i + (y_i - y_{i+1})\omega^{i+1}). \quad (8)$$

In the last system and below all relations are modulo the differential ideal generated by θ^i .

3.2 Exterior derivatives of the new coframe

On the hypothetical integral manifold M^2 , the differentials $d\theta^i$ must also vanish. Direct calculations show

$$d\theta^i = dx^{i+1} \wedge dy^{i+1} - dx^i \wedge dy^i = 0, i \in \mathbb{Z}/4\mathbb{Z}. \quad (9)$$

These identities are related to the area-conservation property of the outer billiard map.

Another calculation gives the relation between some exterior products of the basis elements in the old and new coframes (by taking the exterior product of (7) and (8))

$$dx^{i+1} \wedge dy^{i+1} = -\frac{1}{\Delta_{i,i+1}} \omega^i \wedge \omega^{i+1}. \quad (10)$$

From (9) and (10) we obtain that on M^2 the following relations hold

$$\Delta_{i,i+1}^{-1} \omega^i \wedge \omega^{i+1} = \Delta_{i-1,i}^{-1} \omega^{i-1} \wedge \omega^i \quad (11)$$

for all i .

Now, we compute differentials of ω^i :

$$d\omega^i = \frac{4}{\Delta_i} \omega^i \wedge \omega^{i+1},$$

where we use the notation $\Delta_i := \Delta_{i,i+1}$. Using (11) we conclude that

$$d\omega^i = \frac{4}{\Delta_j} \omega^j \wedge \omega^{j+1}. \quad (12)$$

for any $i, j \in \mathbb{Z}/4\mathbb{Z}$.

3.3 The case of 3-period orbits

Here we reproduce a result from [6] using EDS. In this case $\Delta_{i,i+1}$ is the double area enclosed by the triangular periodic orbit. Then the above relations simplify

$$\omega^i \wedge \omega^{i+1} = \omega^{i-1} \wedge \omega^i \quad (13)$$

$$d\omega^i = \frac{3}{S} \omega^j \wedge \omega^{j+1}, \quad (14)$$

where $i \in \mathbb{Z}/3\mathbb{Z}, j \in \mathbb{Z}/3\mathbb{Z}$ are arbitrary.

Since $\omega^1 \wedge \omega^2 \neq 0$ on M^2 , then we must have a relation

$$\omega^3 = a\omega^1 + b\omega^2.$$

Taking exterior product and using the above relations, we obtain

$$\omega^1 + \omega^2 + \omega^3 = 0$$

and therefore,

$$d\omega^1 + d\omega^2 + d\omega^3 = 0 \Rightarrow 3\omega^1 \wedge \omega^2 = 0$$

contradicting the independence of ω^1, ω^2 .

3.4 Integral elements

On M^2 at most two 1-forms can be linearly independent. Let us assume that $\omega^1 \wedge \omega^3 \neq 0$. The case when $\omega^1 \wedge \omega^3 = 0$ will be considered separately. The remaining 1-forms are then linearly dependent on ω^1, ω^3 :

$$\omega^2 = a_1 \omega^1 + a_3 \omega^3 \quad (15)$$

$$\omega^4 = b_1 \omega^1 + b_3 \omega^3. \quad (16)$$

Taking the exterior product of both equations with ω^2, ω^4 and assuming $\mathcal{D} = \Delta_2\Delta_4 - \Delta_1\Delta_3 \neq 0$ (the case $\mathcal{D} = 0$ will be also evaluated separately), we obtain

$$\begin{aligned} 0 = \omega^2 \wedge \omega^2 &= a_1 \omega^1 \wedge \omega^2 + a_3 \omega^3 \wedge \omega^2 \\ 0 = \omega^4 \wedge \omega^4 &= b_1 \omega^1 \wedge \omega^4 + b_3 \omega^3 \wedge \omega^4 \\ \omega^2 \wedge \omega^1 &= a_3 \omega^3 \wedge \omega^1 \\ \omega^4 \wedge \omega^1 &= b_3 \omega^3 \wedge \omega^1. \end{aligned}$$

Using the relations in (11), we obtain from the first two equations

$$\frac{a_1}{a_3} = \frac{\Delta_2}{\Delta_1} \quad \text{and} \quad \frac{b_3}{b_1} = \frac{\Delta_4}{\Delta_3} \quad (17)$$

and from the last two

$$\frac{a_3}{\Delta_1} + \frac{b_3}{\Delta_4} = 0. \quad (18)$$

Expressing all the coefficients in terms of b_3 and then using the notation $v := -b_3/\Delta_4$, we find the relations

$$\omega^2 = v(\Delta_2\omega^1 + \Delta_1\omega^3) \quad (19)$$

$$\omega^4 = -v(\Delta_3\omega^1 + \Delta_4\omega^3), \quad (20)$$

where v is a function defined on M^2 .

Taking the exterior product of the above two equations, we obtain the relation

$$\omega^2 \wedge \omega^4 = -v^2 \mathcal{D} \omega^1 \wedge \omega^3. \quad (21)$$

To compute dv , we need first to evaluate $d\Delta^i$. Using the definition of Δ_i , the relations between the new and old coframes, and applying the following property

$$\sum_{j=1}^4 \Delta_{i,j} = \sum_{j=1}^4 \begin{vmatrix} x_i - x_{i+1} & y_i - y_{i+1} \\ x_j - x_{j+1} & y_j - y_{j+1} \end{vmatrix} = \sum_{j=1}^4 \begin{vmatrix} x_i - x_{i+1} & y_i - y_{i+1} \\ 0 & 0 \end{vmatrix} = 0,$$

we obtain

$$d\Delta^i = \frac{\Delta_{i+2}}{\Delta_{i-1}} \omega^i - \frac{\Delta_i}{\Delta_{i+1}} \omega^{i+2} + \frac{\Delta_i}{\Delta_{i-1}} \omega^{i-1} - \frac{\Delta_{i+2}}{\Delta_{i+1}} \omega^{i+1}.$$

Using the equations (19–20), we obtain

$$d\Delta^1 = \frac{\Delta_3}{\Delta_4} (1 - v(\Delta_1 + \Delta_4)) \omega^1 + \frac{\Delta_1}{\Delta_2} (-1 - v(\Delta_2 + \Delta_3)) \omega^3 \quad (22)$$

$$d\Delta_2 = \frac{\Delta_2}{\Delta_1} (1 + v(\Delta_1 + \Delta_4)) \omega^1 + \frac{\Delta_4}{\Delta_3} (-1 + v(\Delta_2 + \Delta_3)) \omega^3. \quad (23)$$

Solving for ω^1, ω^3 in (19–20),

$$\omega^1 = \frac{1}{v\mathcal{D}} (\Delta_4\omega^2 + \Delta_1\omega^4) \quad (24)$$

$$\omega^3 = -\frac{1}{v\mathcal{D}} (\Delta_3\omega^2 + \Delta_2\omega^4) \quad (25)$$

and substituting these expressions in (22–23), we obtain

$$d\Delta^1 = \frac{\Delta_3}{\Delta_2\mathcal{D}v} (\Delta_1 + \Delta_2 - \mathcal{D}v) \omega^2 + \frac{\Delta_1}{\Delta_4\mathcal{D}v} (\Delta_3 + \Delta_4 + \mathcal{D}v) \omega^4 \quad (26)$$

$$d\Delta^2 = \frac{\Delta_4}{\Delta_1\mathcal{D}v} (\Delta_1 + \Delta_2 + \mathcal{D}v) \omega^2 + \frac{\Delta_2}{\Delta_3\mathcal{D}v} (\Delta_3 + \Delta_4 - \mathcal{D}v) \omega^4. \quad (27)$$

We can also express ω^2, ω^4 through $d\Delta^1, d\Delta^2$ by inverting the above equations:

$$\omega^2 = - \frac{1}{8S} \left[\frac{\Delta_2}{\Delta_3} (\Delta_3 + \Delta_4 - \mathcal{D}v) d\Delta^1 - \frac{\Delta_1}{\Delta_4} (\Delta_3 + \Delta_4 + \mathcal{D}v) d\Delta^2 \right] \quad (28)$$

$$\omega^4 = \frac{1}{8S} \left[\frac{\Delta_4}{\Delta_1} (\Delta_1 + \Delta_2 + \mathcal{D}v) d\Delta^1 - \frac{\Delta_3}{\Delta_2} (\Delta_1 + \Delta_2 - \mathcal{D}v) d\Delta^2 \right]. \quad (29)$$

3.5 Special solutions

There exist outer billiards in which 4-period orbits constitute a set of positive measure [15]. More precisely, the following statement holds:

Proposition 3.1 *Let z_1, z_2, z_3, z_4 be a convex quadrilateral and let $\zeta_i = (z_i + z_{i+1})/2, i = 1, 2, 3, 4$ be an outer billiard defined by the middle points of the initial quadrilateral. Then, there exist an open neighborhood $O(z_1) \subset \mathbb{R}^2$, containing only 4-period points.*

Proof: By the well known property of the triangle: midsegment between two sides is parallel to the third side, we observe that midpoints ζ_1, \dots, ζ_4 form a parallelogram. Then, z_1, \dots, z_4 is a 4-period orbit in the outer billiard with this boundary. It remains to show that if z is sufficiently close to z_1 then z is a footpoint of a 4-period orbit. It follows easily from the same midsegment theorem assuming that we take small enough neighborhood so that its images do not intersect any of the lines containing the parallelogram sides. □

Therefore, each nondegenerate quadrilateral belongs to a two dimensional integral submanifold (actually it is a linear integral subspace) in the exterior differential system. Consider, the following specific example: let the outer billiard be given by a unit square with vertices in $(0, 0), (0, 1), (1, 0), (1, 1)$. Let $z_1 - (1/2, -i/2)$ be small. Then, the other vertices of periodic orbits are given by

$$z_1 + z_2 = 2, z_2 + z_3 = 2 + i2, z_3 + z_4 = i2, z_1 + z_4 = 0. \quad (30)$$

Solving this linear system, we obtain

$$\begin{aligned} x_4 &= -x_1 & x_2 &= 2 - x_1 & x_3 &= x_1 \\ y_4 &= -y_1 & y_2 &= -y_1 & y_3 &= y_1 + 2. \end{aligned}$$

Using (1) and the definition of Δ_i , it is easy to compute:

$$\begin{aligned} \Delta_1 &= 4(1 - x_1) \\ \Delta_2 &= 4(y_1 + 1) \\ \omega^1 &= 2y_1 dx^1 - 2(x_1 - 1) dy^1 \\ \omega^2 &= 2(y_1 + 1) dx^1 + 2(1 - x_1) dy^1 \\ \omega^3 &= 2(y_1 + 1) dx^1 - 2x_1 dy^1 \end{aligned}$$

and then substituting these expressions in (19), we obtain

$$v = \frac{1}{4(y_1 + 1 - x_1)} \Rightarrow v = \frac{1}{\Delta_2 - \Delta_3}. \quad (31)$$

This calculation shows that for $v = 1/(\Delta_2 - \Delta_3)$ there exists a solution for each quadrilateral¹ and there are no other solutions with such v . Indeed, this modified EDS has 2 additional 1-forms which must vanish and which are linearly independent:

$$\theta^5 = \Delta_2 \omega^1 + (\Delta_3 - \Delta_2) \omega^2 + \Delta_1 \omega^3 \quad (32)$$

$$\theta^6 = \Delta_3 \omega^1 + \Delta_4 \omega^3 + (\Delta_2 - \Delta_3) \omega^4. \quad (33)$$

But then, by the standard ODE argument there is at most one solution.

3.6 Computation of du

Knowing the special solution, it is now convenient to change the parameter

$$v = \frac{u}{\Delta_2 - \Delta_3},$$

so that the special solution corresponds to $u \equiv 1$. Taking exterior derivative of (19) with v replaced by $u/(\Delta_2 - \Delta_3)$

$$(\Delta_2 - \Delta_3) \omega^2 = u(\Delta_2 \omega^1 + \Delta_1 \omega^3), \quad (34)$$

we obtain (using $d\Delta_1 = -d\Delta_3$ and $d\omega^i = d\omega^j$)

$$\begin{aligned} & d(\Delta_1 + \Delta_2) \omega^2 + (\Delta_2 - \Delta_3) d\omega^2 = \\ & \frac{\Delta_2 - \Delta_3}{u} du \wedge \omega^2 + u(d\Delta^2 \wedge \omega^1 + d\Delta^1 \wedge \omega^3) + u(\Delta_2 + \Delta_1) d\omega^2. \end{aligned} \quad (35)$$

Now, using (26-27) and (12), we obtain

$$\begin{aligned} & \left[\frac{\Delta_1}{\Delta_4 \mathcal{D}v} (\Delta_3 + \Delta_4 + \mathcal{D}v) + \frac{\Delta_2}{\Delta_3 \mathcal{D}v} (\Delta_3 + \Delta_4 - \mathcal{D}v) \right] \omega^4 \wedge \omega^2 + d\omega^2 (\Delta_2 - \Delta_3 - u(\Delta_1 + \Delta_2)) = \\ & \frac{\Delta_2 - \Delta_3}{u} du \wedge \omega^2 + u \left[\frac{\Delta_4}{\Delta_3} (-1 + v(\Delta_2 + \Delta_3)) - \frac{\Delta_3}{\Delta_4} (-1 + v(\Delta_1 + \Delta_4)) \right] \omega^3 \wedge \omega^1. \end{aligned}$$

The last expression can be further simplified

$$\mathcal{D}du \wedge \omega^2 + 4(\Delta_1 + \Delta_2 - S)(1 - u) \omega^2 \wedge \omega^4 = 0, \quad (36)$$

¹Indeed, since outer billiard map commutes with affine transformations, similar solution with the same relation (31) exists for arbitrary nondegenerate convex quadrilateral.

where we used:

$$\begin{aligned}\mathcal{D} &= \Delta_2 \Delta_4 - \Delta_1 \Delta_3 = (\Delta_1 - \Delta_2)(\Delta_2 - \Delta_3) \\ \mathcal{D}v &= (\Delta_1 - \Delta_2)u \\ d\omega^i &= \frac{4}{\Delta_2} \omega^2 \wedge \omega^3 = 4v\omega^1 \wedge \omega^3 = -\frac{4}{v\mathcal{D}} \omega^2 \wedge \omega^4 = -\frac{4}{u(\Delta_1 - \Delta_2)} \omega^2 \wedge \omega^4 \\ \Delta_1 + \Delta_3 &= \Delta_2 + \Delta_4 = 2S \text{ (= constant)}.\end{aligned}$$

To derive similar relation for $du \wedge \omega^4$, we add up (19-20)

$$(\Delta_2 - \Delta_3)(\omega^2 + \omega^4) = u((\Delta_2 - \Delta_3)\omega^1 + (\Delta_1 - \Delta_4)\omega^3)$$

and since $\Delta_1 - \Delta_4 = \Delta_2 - \Delta_3$, we have

$$(\Delta_2 - \Delta_3)(\omega^2 + \omega^4) = u(\Delta_2 - \Delta_3)(\omega^1 + \omega^3) \Rightarrow \omega^2 + \omega^4 = u(\omega^1 + \omega^3).$$

Taking exterior derivative, we obtain

$$2d, w^i = du \wedge (\omega^1 + \omega^3) + 2u d\omega^i,$$

which implies

$$-\frac{8}{u(\Delta_1 - \Delta_2)} \omega^2 \wedge \omega^4 = \frac{1}{u} du \wedge (\omega^2 + \omega^4) - \frac{8}{\Delta_1 - \Delta_2} \omega^2 \wedge \omega^4$$

and after multiplying with $u\mathcal{D}$

$$\mathcal{D}du \wedge (\omega^2 + \omega^4) + 8(1-u)(\Delta_2 - \Delta_3)\omega^2 \wedge \omega^4.$$

Subtracting (36) from the last expression, we obtain

$$\mathcal{D}du \wedge \omega^4 + 4(\Delta_2 - \Delta_3 - S)(1-u)\omega^2 \wedge \omega^4 = 0 \quad (37)$$

with (36) rewritten in a similar form:

$$\mathcal{D}du \wedge \omega^2 + 4(\Delta_2 - \Delta_3 + S)(1-u)\omega^2 \wedge \omega^4 = 0. \quad (38)$$

Then

$$\mathcal{D}du = (1-u)((S - \Delta_2 + \Delta_3)\omega^2 + (\Delta_2 - \Delta_3 + S)\omega^4). \quad (39)$$

Substituting (28-29) in the last expression we obtain

$$\frac{8S}{1-u} du = (a_1 u + b_1) d\Delta^1 + (a_2 u + b_2) d\Delta^2, \quad (40)$$

where

$$a_1 = \frac{\Delta_4}{\Delta_1} \left(+1 + \frac{S}{\Delta_2 - \Delta_3} \right) + \frac{\Delta_2}{\Delta_3} \left(-1 + \frac{S}{\Delta_2 - \Delta_3} \right) \quad (41)$$

$$a_2 = \frac{\Delta_1}{\Delta_4} \left(-1 + \frac{S}{\Delta_2 - \Delta_3} \right) + \frac{\Delta_3}{\Delta_2} \left(+1 + \frac{S}{\Delta_2 - \Delta_3} \right) \quad (42)$$

$$b_1 = \frac{1}{\mathcal{D}} \left[\frac{\Delta_4}{\Delta_1} (\Delta_1 + \Delta_2)(S + \Delta_2 - \Delta_3) - \frac{\Delta_2}{\Delta_3} (\Delta_3 + \Delta_4)(S - \Delta_2 + \Delta_3) \right] \quad (43)$$

$$b_2 = \frac{1}{\mathcal{D}} \left[\frac{\Delta_1}{\Delta_4} (\Delta_3 + \Delta_4)(S - \Delta_2 + \Delta_3) - \frac{\Delta_3}{\Delta_2} (\Delta_1 + \Delta_2)(S + \Delta_2 - \Delta_3) \right]. \quad (44)$$

Taking the exterior derivative of (40) we obtain

$$0 = d \left(\frac{8S}{1-u} du \right) = (-(u\partial_2 a_1 + \partial_2 b_1 + a_1 \partial_2 u) + (u\partial_1 a_2 + \partial_1 b_2 + a_2 \partial_1 u)) d\Delta^1 \wedge d\Delta^2, \quad (45)$$

where $\partial_i := \frac{\partial}{\partial \Delta_i}$.

Next, we use the expression for du , to replace $\partial_i u$ by $(1-u)(a_i u + b_i)/8S$:

$$u \left(\partial_2 a_1 - \partial_1 a_2 + \frac{a_2 b_1 - a_1 b_2}{8S} \right) + \partial_2 b_1 - \partial_1 b_2 + \frac{a_2 b_1 - a_1 b_2}{8S} = 0. \quad (46)$$

Using Maple and then some simplifications, we compute

$$\begin{aligned} & \frac{S(u-1)}{\Delta_1 \Delta_2 \Delta_3 \Delta_4 \mathcal{D}} (-\Delta_1^4 + 4\Delta_1^3 S + 5\Delta_1^2 \Delta_2^2 - 10\Delta_1^2 \Delta_2 S - 3\Delta_1^2 S^2 + 20\Delta_1 \Delta_2 S^2 - \\ & 2\Delta_1 S^3 - 10\Delta_1 \Delta_2^2 S - 2S^3 \Delta_2 - \Delta_2^4 + 4\Delta_2^3 S - 3\Delta_2^2 S^2) = 0. \end{aligned}$$

The last equality cannot hold identically on an open subset. Indeed, we have imposed $u \neq 1$ and the numerator is a nontrivial polynomial in two variables and thus, cannot vanish on an open set.

Remark 3.1 *It is interesting to note that the last expression has the form $(u-1)f(\Delta_1, \Delta_2)$. In other words, $u-1$ can be factored out again making the calculations much easier. In general, one might expect this expression to be of the form: $f(\Delta_1, \Delta_2)u + g(\Delta_1, \Delta_2) = 0$. Then we would have to take the exterior derivative once more and then check solvability condition of this new system.*

3.7 Degenerate cases: $\omega^1 \wedge \omega^3 = 0$ or $\mathcal{D} = 0$.

3.7.1 $\omega^1 \wedge \omega^3 = 0$

Lemma 3.1

$$\omega^1 \wedge \omega^3 = 0 \text{ implies } \Delta_1 = \Delta_2 \text{ or } \Delta_1 = \Delta_4. \quad (47)$$

Proof:

Since $\omega^1 \wedge \omega^2 \neq 0$ we can represent integral elements by

$$\omega^3 = a_1\omega^1 + a_2\omega^2 \quad (48)$$

$$\omega^4 = b_1\omega^1 + b_2\omega^2 \quad (49)$$

and then

$$\omega^3 \wedge \omega^1 = a_2\omega^2 \wedge \omega^1 \Rightarrow a_2 = 0$$

and

$$\omega^3 \wedge \omega^2 = a_1\omega^1 \wedge \omega^2.$$

Using the relations (11), we then obtain

$$a_1 = -\frac{\Delta_2}{\Delta_1}.$$

Therefore, we have

$$\Delta_1\omega^3 + \Delta_2\omega^1 = 0.$$

Similarly, we obtain for b_2 ,

$$\omega^4 \wedge \omega^1 = b_2\omega^2 \wedge \omega^1,$$

which with (11) implies

$$b_2 = -\frac{\Delta_4}{\Delta_1}.$$

For b_2 , we also have

$$\omega^4 \wedge \omega^3 = b_2\omega^2 \wedge \omega^3.$$

Then, using (11) once more, we have

$$b_2 = -\frac{\Delta_3}{\Delta_2}.$$

Now, using both equations for b_2 , we obtain

$$\Delta_2\Delta_4 = \Delta_1\Delta_3 \Rightarrow \mathcal{D} = 0.$$

On the other hand, $\mathcal{D} = 0$ implies

$$\mathcal{D} = (\Delta_1 - \Delta_2)(\Delta_1 - \Delta_4) = 0. \quad (50)$$

Therefore, in some neighborhood of M^2 , either $\Delta_1 = \Delta_2$ or $\Delta_1 = \Delta_4$.

□

Suppose, first that $\omega^2 \wedge \omega^4 = 0$, then

$$\omega^1 \wedge \omega^3 = \omega^2 \wedge \omega^4 = 0.$$

Taking exterior products of (48-49) with ω^i and using (11), we obtain

$$\omega^3 = -\frac{\Delta_2}{\Delta_1}\omega^1 \quad (51)$$

$$\omega^4 = -\frac{\Delta_4}{\Delta_1}\omega^2. \quad (52)$$

However, either $\Delta_1 = \Delta_2$ or $\Delta_1 = \Delta_4$. In the first case, we obtain

$$\omega^3 + \omega^1 = 0 \Rightarrow d\omega^3 = -d\omega^1,$$

but this contradicts (12). Similarly, in the second case ($\Delta_1 = \Delta_4$), we obtain $d\omega^4 = -d\omega^2$ also leading to contradiction.

Now, we are left to consider the case $\omega^2 \wedge \omega^4 \neq 0$. By relabeling, this case is equivalent to the case considered in the next section $\mathcal{D} = 0, \omega^2 \wedge \omega^4 = 0, \omega^1 \wedge \omega^3 \neq 0$.

3.8 $\mathcal{D} = 0, \omega^1 \wedge \omega^3 \neq 0$

In this case we can use representation of integral elements (19-20) and then we also have $\omega^2 \wedge \omega^4 = 0$. Using (50) we also have that $\Delta_1 = \Delta_2$ or $\Delta_1 = \Delta_4$. Assume first that

$$\Delta_1 = \Delta_2 \Rightarrow \Delta_3 = \Delta_4. \quad (53)$$

Now, using formulae (22-23), we obtain

$$v(\Delta_1 + \Delta_4) = 0,$$

which can only occur if $v = 0$ on M^2 . Then, by (19-20) two 1-forms vanish identically $\omega^2 = \omega^4 = 0$ implying $\omega^1 \wedge \omega^2 = 0$, which contradicts the genericity assumption.

In the second scenario $\Delta_1 = \Delta_4$, similar calculations lead to the same contradiction.

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